Foundations of Game Engine Development 1: Mathematics

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1 Vector Formulas

$||\mathbf{v}|| = \sqrt{v_x^2 + v_y^2 + v_z^2 + ...}$ Vector Magnitude

$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

$||t\mathbf{v}|| = |t|||\mathbf{v}||$ Scalar magnitude property

$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_x b_z + ...$ Dot product of vectors

$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||||\mathbf{b}||cos\theta$

$\mathbf{a} \cdot \mathbf{a} = a^2$ Notation (indicates a scalar result)

a and **b** are orthogonal (perpindicular) if $\mathbf{a} \cdot \mathbf{b} = 0$

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$ Cross product on 3D vectors

$\mathbf{a} \times \mathbf{b} = ||\mathbf{a}||||\mathbf{b}||cos\theta$

$A = \frac{1}{2}||(\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0)||$ Area of a triangle defined by 3 points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$

$||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||||\mathbf{b}||sin\theta$ Magnitude of a cross product

$\hat{\mathbf{v}} = \frac{\mathbf{v}}{||\mathbf{v}||}$ Normalize a vector (becomes a unit vector, members sum to 1)

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ Vector Triple Product ("back minus cab")

$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$ Scalar triple prodct. Equals the volume of a parallelepiped formed by a,b, and c.

$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}]$

$\mathbf{a}_{||\mathbf{b}} = \frac{\mathbf{a} \cdot \mathbf{b}}{b^2}\mathbf{b}$ Projection of a onto b (cosine/X component)

$$\mathbf{a}_{||\mathbf{b}} = \frac{1}{b^2} \begin{bmatrix} b_x^2 & b_x b_y & b_x b_z \\ b_x b_y & b_y^2 & b_y b_z \\ b_x b_z & b_y b_z & b_z^2 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$ Projection of a onto b with 3d vectors

$\mathbf{a}_{\perp\mathbf{b}} = \mathbf{a} - \mathbf{a}_{||\mathbf{b}}$ Rejection of b from a (sin/Y component)

$(\mathbf{a}_{||\mathbf{b}})^2 + (\mathbf{a}_{\perp\mathbf{b}})^2 = a^2$ Projection/Rejection property (pythagorean theorem)

$\mathbf{u}_1 = \mathbf{v}_1, \mathbf{u}_2 = \mathbf{v}_2 - (\mathbf{v}_2)_{||\mathbf{u}_1}, \mathbf{u}_3 = \mathbf{v}_3 - (\mathbf{v}_3)_{||\mathbf{u}_1} - (\mathbf{v}_3)_{||\mathbf{u}_2}, ...$ Gram-Schmidt process (orthogonal vectors set).

Common to normalize each result (*orthonormalization*)

$$\mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^T = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \begin{bmatrix} b_x & b_y & b_z \end{bmatrix}$$ 3D Vector Outer Product

$$= \begin{bmatrix} a_x b_z & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{bmatrix}$$

$\mathbf{v} = (v_x, v_y, v_z, 0)$ Homogenous direction vector

$\boldsymbol{p} = (v_x, v_y, v_z, 1)$ Homogenous position vector

2 Matrix Formulas

$\mathbf{A}_{n,m}\mathbf{B}_{m,p} = \mathbf{C}_{n,p}$ — Matrix mult result size; also cols A must equal rows B

$\mathbf{M}^T_{i,j} = \mathbf{M}_{j,i}$ — Matrix transpose

$\mathbf{M} = [\mathbf{abc}]$ — Column vector representation

$$\mathbf{M} = \begin{bmatrix} X & a & b \\ a & X & c \\ b & c & X \end{bmatrix}$$

Symmetric matrix (X is be any value)

$\mathbf{M}^T = \mathbf{M}$

$$\mathbf{M} = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

Antisymmetric matrix or skew-symmetric matrix (diagonals must be 0)

$\mathbf{M}^T = -\mathbf{M}$

$\mathbf{M}\mathbf{v} = v_x\mathbf{a} + v_y\mathbf{b} + v_z\mathbf{c}$ — Column vector mult notation (a,b,c cols of M)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

2D Matrix determinant

$det(\mathbf{M}) = \sum_{i=0}^{n-1} M_{ik}(-1)^{i+k}|\mathbf{M}_{\bar{i}\bar{k}}|$ — Recursive determinant (each $|M_{ik}|$ is the sub matrix excluding the current col)

k is chosen as any row; good when row is mostly zeros

$C_{ij}(\mathbf{M}) = (-1)^{i+j}|\mathbf{M}_{\bar{i}\bar{j}}|$ — Cofactor matrix definition

$\mathbf{M}^{-1} = \frac{1}{det(\mathbf{M})}\mathbf{C}^T(\mathbf{M})$ — Inverse; $C^T(M)$ = adjugate matrix

$$\mathbf{A}^{-1} = \frac{1}{A_{00}A_{11} - A_{01}A_{10}} \begin{bmatrix} A_{11} & -A_{01} \\ -A_{10} & A_{00} \end{bmatrix}$$

2D Matrix inverse

$$\mathbf{B}^{-1} = \frac{1}{det(\mathbf{B})} \begin{bmatrix} B_{11}B_{22} - B_{12}B_{21} & B_{02}B_{21} - B_{01}B_{22} & B_{01}B_{12} - B_{02}B_{11} \\ B_{12}B_{20} - B_{10}B_{22} & B_{00}B_{22} - B_{02}B_{20} & B_{02}B_{10} - B_{00}B_{12} \\ B_{10}B_{21} - B_{11}B_{20} & B_{01}B_{20} - B_{00}B_{21} & B_{00}B_{11} - B_{01}B_{10} \end{bmatrix}$$

3D Matrix inverse

$$\mathbf{M}^{-1} = \frac{1}{[\mathbf{a},\mathbf{b},\mathbf{c}]} \begin{bmatrix} \mathbf{b} \times \mathbf{c} \\ \mathbf{c} \times \mathbf{a} \\ \mathbf{a} \times \mathbf{b} \end{bmatrix}$$

3D matrix inverse (a,b,c cols of M)

$$\mathbf{M}^{-1} = \frac{1}{\mathbf{s}\cdot\mathbf{v} + \mathbf{t}\cdot\mathbf{u}} \begin{bmatrix} \mathbf{b} \times \mathbf{v} + y\mathbf{t}| - \mathbf{b}\cdot\mathbf{t} \\ \mathbf{v} \times \mathbf{a} - x\mathbf{t}|\mathbf{a}\cdot\mathbf{t} \\ \mathbf{d} \times \mathbf{u} + w\mathbf{s}| - \mathbf{d}\cdot\mathbf{s} \\ \mathbf{u} \times \mathbf{c} - z\mathbf{s}|\mathbf{c}\cdot\mathbf{s} \end{bmatrix}$$

4D Matrix inverse. a,b,c,d = 3D column vectors of M.

x,y,z,w = last row of M, x = $a \times b$, t = $c \times d$, u = $ya - xb$, v = $wc - zd$

3 Transformations

$\mathbf{Mv} = v_x\mathbf{a} + v_y\mathbf{b} + v_z\mathbf{c}$

Where a,b,c are the columns of M

$$\mathbf{M}^T\mathbf{M} = \begin{bmatrix} a^2 & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & b^2 & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & c^2 \end{bmatrix}$$

Orthogonal transform; a,b,c are cols of M. Unit cols and all perpindicular. $M^T = M^-1$

$\mathbf{B} = \mathbf{MAM}^{-1}$

Transform from coord space A applied in coord space B. M = coord space transform.

$$M_{rot_x}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}$$

Rotation matrix about X axis

$$M_{rot_y}(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{bmatrix}$$

Rotation matrix about Y axis

$$M_{rot_z}(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation matrix about Z axis

$M_{rot}(\theta, \mathbf{a}) =$

$$\begin{bmatrix} c + (1-c)a_x^2 & (1-c)a_xa_y - sa_z & (1-c)a_xa_z + sa_y \\ (1-c)a_xa_y + sa_z & c + (1-c)a_y^2 & (1-c)a_ya_z - sa_x \\ (1-c)a_xa_z - sa_y & (1-c)a_ya_z + sa_x & c + (1-c)a_z^2 \end{bmatrix}$$

Rotation matrix about arbitrary axis **a**. c = $cos\theta$, s = $sin\theta$

$M_{reflect}(\mathbf{a}) =$

$$\begin{bmatrix} 1 - 2a_x^2 & -2a_xa_y & -2a_xa_z \\ -2a_xa_y & 1 - 2a_y^2 & -2a_ya_z \\ -2a_xa_z & -2a_ya_z & 1 - 2a_z^2 \end{bmatrix}$$

Reflection matrix about axis **a**, assumes a unit length.

$M_{invol}(\mathbf{a}) =$

$$\begin{bmatrix} 2a_x^2 - 1 & 2a_xa_y & 2a_xa_z \\ 2a_xa_y & 2a_y^2 - 1 & 2a_ya_z \\ 2a_xa_z & 2a_ya_z & 2a_z^2 - 1 \end{bmatrix}$$

Involution matrix about axis **a**, assumes a unit length. Negation of reflect mat.

$$M_{scale}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}$$

Scale matrix about x,y,z axes

$M_{scale}(s, \mathbf{a}) =$

$$\begin{bmatrix} (s-1)a_x^2 + 1 & (s-1)a_xa_y & (s-1)a_xa_z \\ (s-1)a_xa_y & (s-1)a_y^2 + 1 & (s-1)a_ya_z \\ (s-1)a_xa_z & (s-1)a_ya_z & (s-1)a_z^2 + 1 \end{bmatrix}$$

Scale matrix along direction **a**

$M_{skew}(\theta, \mathbf{a}, \mathbf{b}) =$ Skew matrix along direction **a** based on length projection **b**

$$\begin{bmatrix} a_x b_x tan\theta + 1 & a_x b_y tan\theta & a_x b_z tan\theta \\ a_y b_x tan\theta & a_y b_y tan\theta + 1 & a_y b_z tan\theta \\ a_z b_x tan\theta & a_z b_y tan\theta & a_z b_z tan\theta + 1 \end{bmatrix}$$

$M_{skew}(\theta, \mathbf{a}, \mathbf{b}) = \mathbf{I} + tan\theta(\mathbf{a} \otimes \mathbf{b})$ Alternative skew matrix formation

$\mathbf{H} = \begin{bmatrix} \mathbf{M} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$ Homogenous transformation matrix; M = 3x3 transf mat, t = translate vec

$\mathbf{H}^{-1} = \begin{bmatrix} \mathbf{M}^{-1} & -\mathbf{M}^{-1}\mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$ Homogenous transformation matrix inverse

4 Quaternions

$\mathbf{q} = xi + yj + zk + w$ Quaternion rep; i,j,k imaginary values, w is scalar val

$\mathbf{q} = \mathbf{v} + s$ alt rep. v = xyz, s = w scalar

$\mathbf{q} = (sin\frac{\theta}{2})\mathbf{a} + cos\frac{\theta}{2}$ rep. where rotation of θ about axis **a**

$\mathbf{q}_1\mathbf{q}_2 = \mathbf{v}_1 \times \mathbf{v}_2 + s_1\mathbf{v}_2 + s_2\mathbf{v}_1 + s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2$ Quaternion multiplication

$\mathbf{q}_2\mathbf{q}_1 = \mathbf{q}_1\mathbf{q}_2 - 2(\mathbf{v}_1 \times \mathbf{v}_2)$ Order of multiplication property

$\mathbf{q}^* = -\mathbf{v} + s$ Quaternion conjugate

$\mathbf{q}\mathbf{q}^* = \mathbf{q}^*\mathbf{q} = v^2 + s^2$ Conjugate multiply

$||\mathbf{q}|| = \sqrt{\mathbf{q}\mathbf{q}^*} = \sqrt{v^2 + s^2}$ Quaternion magnitude

$\mathbf{q}^{-1} = \frac{\mathbf{q}^*}{\mathbf{q}\mathbf{q}^*} = \frac{-\mathbf{v}+s}{v^2+s^2}$ Quaternion inverse

$\mathbf{v}^{`} = \mathbf{q}\mathbf{v}\mathbf{q}^{-1}$ Vector rotation using quaternion (uses quat. mult.) v is quaternion of form $v_x i + v_y j + v_z k + 0$.

$\mathbf{v}^{`} = \mathbf{q}\mathbf{v}\mathbf{q}^*$ Vector rotation using quaternion if quat is unit (magnitude = 1)

$\mathbf{v}^{`} = (\mathbf{q}_2\mathbf{q}_1)\mathbf{v}(\mathbf{q}_2\mathbf{q}_1)^*$ Multiple rotations combined

$\mathbf{M}_{rot}(\mathbf{q}) =$ Convert quaternion to rotation matrix

$$\begin{bmatrix} 1 - 2y^2 - 2z^2 & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & 1 - 2x^2 - 2z^2 & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & 1 - 2x^2 - 2y^2 \end{bmatrix}$$

5 Geometry

$\mathbf{n} = (\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0)$ — Outward facing normal vector from points p0,p1,p2

$\mathbf{n}^B = \mathbf{n}^A \mathbf{M}^{-1}$ — transform normal from space A to B with transform matrix M

$\mathbf{n}^B = \mathbf{n}^A adj(\mathbf{M}) = \mathbf{n}^A det(\mathbf{M})\mathbf{M}^{-1}$ — transform a normal formed from cross prodect from space A to B

$d = \sqrt{u^2 - \frac{(\mathbf{u}\cdot\mathbf{v})^2}{v^2}}$ — Distance from point q to line v, u = q-p, p = point on line

$d = \sqrt{\frac{(\mathbf{u}\times\mathbf{v})^2}{v^2}}$ — Alternate distance formula

$\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \frac{1}{(\mathbf{v}_1\cdot\mathbf{v}_2)^2 - v_1^2 v_2^2} \begin{bmatrix} -v_2^2 & \mathbf{v}_1\cdot\mathbf{v}_2 \\ -\mathbf{v}_1\cdot\mathbf{v}_2 & v_1^2 \end{bmatrix} \begin{bmatrix} (\mathbf{p}_2 - \mathbf{p}_1)\cdot\mathbf{v}_1 \\ (\mathbf{p}_2 - \mathbf{p}_1)\cdot\mathbf{v}_2 \end{bmatrix}$ — Time parameters for distance between to parameteric lines at points $p_1, p_2 (L1(t) = p_1 + t_1 v_1)$

$d = ||L_2(t_2) - L_1(t_1)||$ — Distance obtained from the above time parameters between two parametric lines

$d = \sqrt{\frac{[(\mathbf{p}_2 - \mathbf{p}_1)\times\mathbf{v}_1]^2}{v_1^2}}$ — Distance between two lines if they are parallel (determinant = 0)

$\mathbf{f}\cdot\mathbf{p} = 0, \mathbf{f} = [n_x n_y n_z d] = [\mathbf{n}|d]$ — implicit plane representation, **n** is the normal to the plane, d = distance from plane to origin

$d = \mathbf{f}\cdot\mathbf{p}$ — distance d from point **p** to plane **f**

$\mathbf{p}' = \mathbf{p} - 2(\mathbf{f}\cdot\mathbf{p})\mathbf{n}$ — Reflection of point **p** through normalized plane **f**

$\mathbf{H}_{reflect}(\mathbf{f}) = \begin{bmatrix} 1 - 2n_x^2 & -2n_x n_y & -2n_x n_z & -2n_x d \\ -2n_x n_y & 1 - 2n_y^2 & -2n_y n_z & -2n_y d \\ -2n_x n_z & -2n_y n_z & 1 - 2n_z^2 & -2n_z d \\ 0 & 0 & 0 & 1 \end{bmatrix}$ — Reflection matrix through plane **f**

$\mathbf{q} = \mathbf{p} - \frac{\mathbf{f}\cdot\mathbf{p}}{\mathbf{f}\cdot\mathbf{v}}\mathbf{v}$ — Intersection point of a line L(t) = **p** + t**v** with plane **f**

$\mathbf{p} = \frac{d_1(\mathbf{n}_3\times\mathbf{n}_2) + d_2(\mathbf{n}_1\times\mathbf{n}_3) + d_3(\mathbf{n}_2\times\mathbf{n}_1)}{[\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3]}$ — Intersection point of 3 planes (divisor is scalar triple product)

$\mathbf{p} = \frac{d_1(\mathbf{v}\times\mathbf{n}_2) + d_2(\mathbf{n}_1\times\mathbf{v})}{v^2}$ — Intersection point of two planes, $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$

$\mathbf{f}^B = \mathbf{f}^A det(\mathbf{H})\mathbf{H}^{-1} = \mathbf{f}^A adj(\mathbf{H})$ — Transformation of plane f in space A to space B with transform mat H

$\{\mathbf{v}|\mathbf{m}\}, \mathbf{m} = \mathbf{p}_1 \times \mathbf{p}_2$ — Plucker coords rep of a line, v = direction, and p1,p2 are any points on the line (called 'moment')

$(\mathbf{p}|w)$ — Plucker coords rep of a 4D point vector with w component

$d = \frac{|\mathbf{v}_1\cdot\mathbf{m}_2 + \mathbf{v}_2\cdot\mathbf{m}_1|}{||\mathbf{v}_1\times\mathbf{v}_2||}$ — Distance between two lines in Plucker rep

$\{\mathbf{v}^B \mid \mathbf{m}^B\} = \{\mathbf{M}\mathbf{v}^A \mid \mathbf{m}^A adj(\mathbf{M}) + \mathbf{t} \times (\mathbf{M}\mathbf{v}^A)\}$	Line transform from mat H, M = upper 3x3 matrix from H, t = last column (translation) of H

6 Grassman Algebra

$\mathbf{a} \wedge \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{e}_{23} + (a_z b_x - a_x b_z)\mathbf{e}_{31} + (a_x b_y - a_y b_x)\mathbf{e}_{12}$	Wedge product (bivector); $\mathbf{e}_i \wedge \mathbf{e}_j = \mathbf{e}_{ij}$
$\mathbf{a} \vee \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{e}_1 + (a_z b_x - a_x b_z)\mathbf{e}_2 + (a_x b_y + a_y b_x)\mathbf{e}_3$	Antiwedge product
$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$	Anti-commutative wedge property
$\mathbf{a} \wedge \mathbf{a} = 0$	Wedge product zero property
$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = (a_x b_y c_z + a_y b_z c_x + a_z b_x c_y - a_x b_z c_y - a_y b_x c_z - a_z b_y c_x)(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)$	Triple wedge product (trivector)
$gr(\mathbf{A} \wedge \mathbf{B}) = gr(\mathbf{A}) + gr(\mathbf{B})$	Grade property (gr); A and B are k-vectors (e.g. bi or tri) where k is the grade
$gr(\mathbf{A} \wedge \mathbf{B}) = -1^{gr(\mathbf{A})gr(\mathbf{B})}(\mathbf{B} \wedge \mathbf{A})$	Negation commutative property
$\mathbf{E}_n = \mathbf{e}_{12...n} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge ...\mathbf{e}_n$	Unit volume element for a n-dim grassman alg
$\mathbf{A} \wedge \overline{\mathbf{A}} = \mathbf{E}_n$	n-vector complement (contains all basis elements NOT present in original element $\mathbf{A}$)
$\mathbf{a} \times \mathbf{b} = \overline{\mathbf{a} \wedge \mathbf{b}}$	Cross product and wedge product complement
$\underline{\mathbf{A}} = (-1)^{k(n-k)}\overline{\mathbf{A}}$	Left complement when n is even
$\overline{\mathbf{A} \wedge \mathbf{B}} = \overline{\mathbf{A}} \vee \overline{\mathbf{B}}$	Complement property (and similar for all variations, like DeMorgan's rules)
$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \vee \overline{\mathbf{b}}$	Dot product (interior product)
$\mathbf{p} \wedge \mathbf{q} = (q_x - p_x)\mathbf{e}_{41} + (q_y - p_y)\mathbf{e}_{42} + (q_z - p_z)\mathbf{e}_{43} + (p_y q_z - p_z q_y)\mathbf{e}_{23} + (p_z q_x - p_x q_z)\mathbf{e}_{31} + (p_x q_y - p_y q_x)\mathbf{e}_{12}$	Line representation from homogenous points p,q (w = 1)
$\mathbf{p} \wedge \mathbf{L} = (L_{vy} p_z - L_{vz} p_y + L_{mx})\overline{\mathbf{e}_1} + (L_{vz} p_x - L_{vx} p_z + L_{my})\overline{\mathbf{e}_2} + (L_{vx} p_y - L_{vy} p_x + L_{mz})\overline{\mathbf{e}_3} + (-L_{mx} p_x - L_{my} p_y - L_{mz} p_z)\overline{\mathbf{e}_4}$	Plane rep from 3 homogeneous points p,q,r, L = $q \wedge r$ in plucker coords rep ($\{v \mid m\}$)
$\mathbf{f} \vee \mathbf{g} = (f_z g_y - f_y g_z)\mathbf{e}_{41} + (f_x g_z - f_z g_x)\mathbf{e}_{42} + (f_y g_x - f_x g_y)\mathbf{e}_{43} + (f_x g_w - f_w g_x)\mathbf{e}_{23} + (f_y g_w - f_w g_y)\mathbf{e}_{31} + (f_z g_w - f_w g_z)\mathbf{e}_{12}$	plane intersection (a line) of planes f,g
$\mathbf{f} \vee \mathbf{L} = (L_{my} f_z - L_{mz} f_y + L_{vx} f_w)\mathbf{e}_1 + (L_{mz} f_x - L_{mx} f_z + L_{vy} f_w)\mathbf{e}_2 + (L_{mx} f_y - L_{my} f_x + L_{vz} f_w)\mathbf{e}_3 + (-L_{vx} f_x - L_{vy} f_y - L_{vz} f_z)\mathbf{e}_4$	Intersection of plane f, line L (a point)
$d = \frac{\mathbf{L}_1 \vee \mathbf{L}_2}{\vert\vert\mathbf{v}_1 \wedge \mathbf{v}_2\vert\vert}$	Distance between lines L1,L2 ($L = \{v \mid m\}$)
$d = \frac{\mathbf{p} \vee \mathbf{f}}{\vert\vert\mathbf{n}\vert\vert}$	Distance between point p and plane f ($f = \{\mathbf{n} \mid d\}$)

$$[\mathbf{abc}]^{-1} = \frac{1}{\underline{\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}}} \begin{bmatrix} \underline{\mathbf{b}\wedge\mathbf{c}} \\ -\underline{\mathbf{a}\wedge\mathbf{c}} \\ \underline{\mathbf{a}\wedge\mathbf{b}} \end{bmatrix} \qquad \text{Matrix inverse with col vecs a,b,c}$$

$$[\mathbf{abcd}]^{-1} = \frac{1}{\underline{\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}}} \begin{bmatrix} \underline{\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}} \\ -\underline{\mathbf{a}\wedge\mathbf{c}\wedge\mathbf{d}} \\ \underline{\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{d}} \\ -\underline{\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}} \end{bmatrix} \qquad \text{Matrix inverse with col vecs a,b,c,d}$$